

Dyonic Wormholes in 5D Kaluza-Klein Theory

Chiang-Mei Chen*

Department of Physics, National Taiwan University, Taipei 106, Taiwan, R.O.C.

New spherically symmetric dyonic solutions, describing a wormhole-like class of spacetime configurations in five-dimensional Kaluza-Klein theory, are given in an explicit form. For this type of solution the electric and magnetic fields cause a significantly different global structure. For the electric dominated case, the solution is everywhere regular but, when the magnetic strength overcomes the electric contribution, the mouths of the wormhole become singular points. When the electric and magnetic charge parameters are identical, the throats “degenerate” and the solution reduces to the trivial embedding of the four-dimensional massless Reissner-Nordström black hole solution. In addition, their counterparts in eleven-dimensional supergravity are constructed by a non-trivial uplifting.

PACS number(s): 04.20.Jb, 04.50.+h

I. INTRODUCTION

After the brilliant insight of Kaluza and Klein [1] realizing that the Einstein gravity and Maxwell electromagnetism theories can be unified in a five-dimensional manifold, the Kaluza-Klein theory, essentially the five-dimensional pure general relativity, has developed explosively. (A comprehensive discussion of Kaluza-Klein theory is given in a recent review article [2].) This is the first theory advocating the idea that the physical world may have more than four dimensions; it laid a foundation for modern developments such as superstrings and M-theory.

The exact solutions, especially spherically symmetric ones, of Kaluza-Klein theory have been studied extensively. A subset of such solutions corresponds to black holes. Well-known examples include the electric pp-wave obtained by Dobiasch and Maison [3] and the magnetic GPS monopole of Gross, Perry [4] and Sorkin [5]. Later, Gibbons and Wiltshire [6] successfully unified the pp-wave and GPS monopole into a dyonic solution which recently has been generalized, by Rasheed [7], to include a rotation. Furthermore, the hidden symmetry, $SL(3, R)$, of the vacuum 5D metric with two commuting isometries and its solution generating application were analysed in [8–10] and were extended in [11].

In addition to the black holes, there is another subset of solutions, the so-called *wormholes*, which were introduced originally by Wheeler [12]. The most physical interesting property of wormholes is that this type of solution provides a possible, theoretically at the moment, way for time traveling, which, if it can be realized, may lead to a break down of some traditional concepts of nature, especially causality [13] (see [14] and references therein).

The discovery of wormholes for Kaluza-Klein theory dated back to the work done by Chodos and Detweiler [15]. In that paper, a class of regular, spherically symmetric and asymptotically flat solutions characterized by three parameters (mass, electric and scalar charges) was constructed and certain cases were interpreted as wormholes. Afterwards, this class of solutions was generalized to axisymmetric multi-wormholes [16] and to higher dimensions [17] by Clément. More recently, a class of wormholes with diagonal 5D metric was also discussed in [18].

The purpose of present paper is giving a dyonic extension of the massless electric wormhole solutions given in [15–17] and rediscovered recently by Dzhunushaliev [19]. Surprisingly, the exact form of these dyonic wormholes is elegant and succinct. The primary properties of these dyonic wormholes are discussed. In addition, their eleven-dimensional supergravity counterparts are presented.

*E-mail: cmchen@phys.ntu.edu.tw

For a general spherically symmetric dyon characterized by two parameters q (electric) and p (magnetic), we take the following general form for the metric

$$ds_5^2 = \frac{B}{A} (d\chi + \omega dt + 2p \cos\theta d\varphi)^2 - \frac{\Delta}{B} dt^2 + A \left(\frac{dr^2}{\Delta} + d\Omega^2 \right), \quad (1)$$

where χ is the extra fifth coordinate, $d\Omega^2 := d\theta^2 + \sin^2\theta d\varphi^2$ and A, B, ω, Δ are functions which depend only on the variable r . The value of coordinate r is extended to $\{-\infty, \infty\}$.

Regarding the wormhole solutions, the pure electric case found in [15–17, 19] can be rewritten, by choosing appropriate coordinates and discarding a dummy parameter, in a more succinct form which satisfies the Eq. (1) pattern together with the condition $p = 0$ and the following specific functions

$$\text{electric:} \quad \omega_e = \frac{2qr}{r^2 - q^2}, \quad A_e = \Delta_e = r^2 + q^2, \quad B_e = r^2 - q^2. \quad (2)$$

Hereafter, we use the subscripts e, m and d to denote the electric, magnetic and dyonic solutions respectively.

Obviously, the likely “singularities” of above spacetime configuration (2) can locate at $r^2 = q^2$ which divide the entire spacetime into three different regions: two *outer regions* $r^2 > q^2$ (two slices, $r < -|q|$ and $r > |q|$) and one *core region* $r^2 < q^2$, $(-|q| < r < |q|)$. Furthermore, it is easy to recognize that in the two outer regions are asymptotically flat, but in the core region, the t coordinate changes its sign and becomes space-like [20] while the elsewhere space-like fifth coordinate χ changes to time-like. It was shown in [15] that the solution (2) is regular everywhere for $r \in \{-\infty, \infty\}$, so that the five-dimensional geometry is of the Lorentzian wormhole type. However, as pointed out in [21] this wormhole, with only a pure electric charge, is non-traversable in the sense that a physical (non-tachyonic) test particle cannot go from one asymptotic flat region ($r \rightarrow \infty$) to the other ($r \rightarrow -\infty$). The “fake” singularities located at $r^2 = q^2$ are just two mouths of the throat of the wormhole and the area of each mouth is finite, $4\pi q^2$. The detailed analysis, including geometrical properties and stability of these solutions, has been given in [23].

Unfortunately, the exact magnetic solution is “unknown” in the literature, instead of which Dzhunushaliev and Singleton gave a numerical analysis [22] to discuss its expected behavior. The authors claimed that this magnetic solution is a finite *flux tube*, which may provide a reason why free monopoles do not appear to exist in nature: they are confined into monopole-anti monopole pairs in a finite, flux tube-like spacetime that is similar to the flux tube confinement picture of quarks in QCD.

However, every Kaluza-Klein magnetic solution, including the GPS monopole and, of course, this desired new type, can be derived by S -duality from its electric dual partner. (There is a lot of literature discussing this duality, see e.g. [24]). Applying S -duality, the “dual” solution of (2) can be found easily without solving the field equations. Its exact form is of the form (1) along with $q = 0$ and

$$\text{magnetic:} \quad \omega_m = 0, \quad A_m = r^2 - p^2, \quad B_m = \Delta_m = r^2 + p^2. \quad (3)$$

According to (3), the possible singularities could occur at $r^2 = p^2$ and the two outer regions $r^2 > p^2$ are also flat asymptotically. However, in the core region $r^2 < p^2$ all space-like coordinates change their sign and the signature of this region is “all minus” which indicates that the spacetime is not Lorentzian anymore but pseudo-Euclidean. Nevertheless, an apparently wormhole-like (pseudo-Euclidean) configuration, but not a flux tube as claimed in [22], may again be formed in the magnetic solution. However, the geometrical structure of these magnetic solutions is significantly different from the electric ones. The two-surfaces of each “end”, located at $r = \pm p$, of the core region for these magnetic solutions, covered by the coordinates θ and φ , shrink to a point for all values of the magnetic charge. We have checked that the magnetic solutions are indeed singular at $r = \pm p$ (the $5D$ Kretschmann invariant, $K_5 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, is divergent at these points), so that these solution actually do not form wormholes since their spacetime can not be extended analytically from $r \rightarrow -\infty$ to $r \rightarrow \infty$. In general, with more than one time-like dimension, one opens up the possibility of closed time-like curves and causality violations. However, since the region of the “throat” seems to be “pinched off” from the asymptotic regions it probably does not cause any mischief.

It is worth noting that our purely magnetic solution is just the Euclideanized massless Taub-NUT solution [25] with a trivial time direction¹.

¹I am grateful to a referee for pointing out this analogue.

The next task, naturally, is trying to construct a two-parameter dyon which can combine the above two single-parameter electric wormhole and magnetic singular solutions. With the help of the symbolic calculation package *GRG* [26], we have obtained this kind of dyonic solution. The result, written in the form of Eq. (1), is

$$\textbf{dyonic:} \quad \omega_d = \frac{2qr}{r^2 + p^2 - q^2}, \quad A_d = r^2 - p^2 + q^2, \quad B_d = r^2 + p^2 - q^2, \quad \Delta_d = r^2 + p^2 + q^2. \quad (4)$$

We now summarize the essential properties, from the five-dimensional point of view, of these new dyons:

- The solutions (4) have a remarkably elegant and symmetric expression, even simpler than the well-known dyonic black holes in [6].
- For the *electric dominated* case, $q^2 > p^2$, this solution is regular everywhere, which can primarily be verified by the associated Kretschmann invariant.² With extended value of $r \in \{-\infty, \infty\}$, it describes the configuration of a Lorentzian wormhole connecting two asymptotically flat spacetimes. The area of the mouths is finite: $4\pi(q^2 - p^2)$.
- For the *magnetic dominated* case, $p^2 > q^2$, the core region is pseudo-Euclidean and the “mouths” at $r = \pm\sqrt{p^2 - q^2}$ compress to singular points. Actually, these two points are singular thus the geometry can not be extended analytically from one Lorentzian asymptotical flat spacetime to the other through the core.
- For the case $q = \pm p$, the throat degenerates to a point at $r = 0$ and the Kaluza-Klein dyonic solutions are just the trivial embedding of the four-dimensional massless Reissner-Nordström black hole solutions of the Einstein-Maxwell theory.

III. FOUR-DIMENSIONAL OBSERVER’S EXAMINATION

There is a physically important question to ask: What will be experienced for a four-dimensional observer inhabiting the spacetime described by (4)? In order to be consistent with Einstein’s theory of gravity, the four-dimensional effective action reduced from the five-dimensional general relativity should be presented in the Einstein frame forming the so-called Einstein-Maxwell-dilaton theory with a certain dilaton coupling constant (minimal coupling of the scalar field). The $4D$ metric of (1) in the Einstein frame is given by

$$ds_4^2 = -\frac{\Delta}{\sqrt{AB}}dt^2 + \sqrt{AB}\left(\frac{dr^2}{\Delta} + d\Omega^2\right). \quad (5)$$

Therefore, in the Einstein frame, the reduced metric (5) of dyonic solutions (4) shows:

- The distinctness between electric, q , and magnetic, p , parameters disappears. (Since $\Delta_d = r^2 + p^2 + q^2$ and $\sqrt{A_d B_d} = \sqrt{r^4 - (p^2 - q^2)^2}$, thus the metric (5) is invariant under exchange parameters p and q for dyonic solutions (4).)
- The asymptotical limit, $r^2 \rightarrow \infty$, is just the dyonic massless Reissner-Nordström solution,

$$ds_4^2 \sim -\left(1 + \frac{p^2 + q^2}{r^2}\right)dt^2 + \left(1 + \frac{p^2 + q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega. \quad (6)$$

- The two-surface area of “origin” for the electromagnetic charges at $r^2 = |p^2 - q^2|$ is always zero.
- The core region $r^2 < |p^2 - q^2|$ physically does not exist, or it is “invisible”, due to the fact that $\sqrt{A_d B_d}$ is ill-defined in this region. From the dimensional reduction point of view, in this region, the KK reduction along the direction ∂_χ is not valid since the Killing vector becomes time-like and presumably does not have closed orbits.

²A more rigorous way to observe the regularity is by the fact that the determinant of the five-dimensional metric, $|g_5| = A_d^2 \sin^2 \theta$, is non-vanishing for all values of r when $q^2 > p^2$. Therefore, one can, for instance, everywhere locally transform to a frame where the metric tensor is diagonal with finite and non-vanishing elements.

Synthesizing the above arguments, one can claim that within the dyonic solutions (4) a four-dimensional observer will “see” only one of the two regions of spacetime generated by electromagnetic “point-like” sources. Nevertheless, for the four-dimensional observer these charges are entirely “disconnected”. From the five-dimensional point of view, however, the explanation may be complete different — the charges could be connected by a wormhole. Thus, our solutions provide a possible realization of the model of “charge without charge” idea for electromagnetic sources proposed by Wheeler.

IV. COUNTERPARTS IN ELEVEN-DIMENSIONAL SUPERGRAVITY

One can show [27,28] that there exists a “duality” between eight-dimensional vacuum configurations possessing two commuting space-like Killing vectors and eleven-dimensional supergravity solutions satisfying a certain ansatz. Applying this correspondence to our general ansatz of metric (1) smeared to eight dimensions, one can obtain the related counterparts within the framework of M-theory. The solutions are

$$ds_{11}^2 = \left(\frac{B}{A}\right)^{-2/3} (dx_1^2 + dx_2^2) + \left(\frac{A}{B}\right)^{-1/3} \left(\sum_{i=3}^7 dx_i^2\right) + \left(\frac{B}{A}\right)^{1/3} \left[-\frac{\Delta}{B} dt^2 + A \left(\frac{dr^2}{\Delta} + d\Omega^2\right)\right], \quad (7)$$

$$\hat{A}_{tx_1x_2} = \omega, \quad \hat{A}_{\varphi x_1x_2} = 2p \cos \theta. \quad (8)$$

Considering only the pure electric solutions, the eleven-dimensional metric and form field reduce to

$$ds_{11}^2 = \left(\frac{r^2 - q^2}{r^2 + q^2}\right)^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + \left(\frac{r^2 - q^2}{r^2 + q^2}\right)^{1/3} \left[dr^2 + (r^2 + q^2)d\Omega^2 + \sum_{i=3}^7 dx_i^2\right], \quad (9)$$

$$\hat{A}_{tx_1x_2} = \frac{2qr}{r^2 - q^2}. \quad (10)$$

Thus, this is another type of 2-brane which is different from the already known $M2$ -brane and $M2$ -fluxbrane to the eleven-dimensional supergravity. The proper terminology for the above solution intuitively should be “ $M2$ -wormbrane”. Similarly, the counterparts of the magnetic and dyonic solutions are a new type of 5-brane and $2\cup 5$ -brane [27,28]. What physical role may be played by these solutions is still unclear and needs further investigation.

V. CONCLUSION

In this paper, we obtain new spherically symmetric dyonic solutions including wormholes and configurations with naked singularities in the five-dimensional Kaluza-Klein theory. The geometrical structure is determined by the relative strength of the electric and magnetic charges. When the electric charge dominates, i.e. $q^2 > p^2$, the solution is a Lorentzian wormhole and the mouths of the wormhole have finite area. Where, when the magnetic charge dominates, $p^2 > q^2$, the core region is pseudo-Euclidean and its ends shrink to singular points. For this case, the spacetime can not be extended analytically from $r \rightarrow -\infty$ to $r \rightarrow \infty$. Moreover, if the electric and magnetic fields are in “balance”, $p^2 = q^2$, the core degenerates to a point and the solution is just a trivial embedding of $4D$ massless Reissner-Nordström black holes.

However, a four-dimensional observer cannot detect the existence of the core region but rather sees a spacetime generated by a point-like electromagnetic source. Therefore, these wormhole-like solutions show that the Wheeler’s “charge without charge” for the origin of electric and magnetic charges may can be realized in higher dimensions.

It is worth noting that there is an one-parameter flux tube solution which was given in [29,22], explicitly

$$\textbf{Flux tube:} \quad \omega_F = \frac{r}{p}, \quad A_F = B_F = 2p^2, \quad \Delta_F = r^2 + 2p^2. \quad (11)$$

This solution was expected to be an extreme limit of a *dyonic solution* which “can” combine an electric wormhole and a magnetic flux tube. However, in this paper we have shown that the dyonic extension from the electric wormhole couples to the magnetic singular solutions but there is no flux tube. Moreover, the solution (11) can easily be understood as belonging to another category of solutions since it is not asymptotic flat. Therefore, there should exist a generalized *dyonic flux tube* the extreme case of which is just the solution (11).

The author would like to thanks G. Clément, V. Dzhunushaliev, D.V. Gal'tsov, J.M. Nester and D. Singleton for many stimulating discussions and helpful comments in preparing this article. This work is supported by the Taiwan CosPA project.

-
- [1] T. Kaluza, *On the Unification Problem in Physics*, *Sitzungsberichte of the Prussian Ache. of Sci.* (1921) 966; O. Klein, *Quantum Theory and Five-Dimensional Theory of Relativity*, *Z. Phys.* **37** (1926) 895; English translation can be found in: *An Introduction to Kaluza-Klein Theories*, Ed. H.C. Lee (World Scientific, Singapore, 1983).
 - [2] J. M. Overduin and P. S. Wesson, *Kaluza-Klein Gravity* *Phys. Rep.* **283** (1997) 303-380; [gr-qc/9805018](#).
 - [3] P. Dobiash and D. Maison, *Stationary, Spherically Symmetric Solutions of Jordan's Unified Theory of Gravity and Electromagnetism*, *Gen. Rel. Grav.* **14** (1982) 231-242.
 - [4] D. J. Gross and M. J. Perry, *Magnetic Monopoles in Kaluza-Klein Theories*, *Nucl. Phys.* **B226** (1983) 29-48.
 - [5] R. D. Sorkin, *Kaluza-Klein Monopole*, *Phys. Rev. Lett.* **51** (1983) 87-90.
 - [6] G. Gibbons and D. Wiltshire, *Black Holes in Kaluza-Klein Theory*, *Ann. Phys.* **167** (1986) 201-223; Erratum: *ibid* **176** (1987) 393.
 - [7] D. Rasheed, *The Rotating Dyonic Black Holes of Kaluza-Klein Theory*, *Nucl. Phys.* **B454** (1995) 379-401; [hep-th/9505038](#).
 - [8] D. Maison, *Ehlers-Harrison Type Transformations for Jordan's Extended Theory of Gravitation*, *Gen. Rel. Grav.* **10** (1979) 717-723.
 - [9] G. Clément, *Stationary Solutions in Five-Dimensional General Relativity*, *Gen. Rel. Grav.* **18** (1986) 137-160.
 - [10] G. Clément, *Solutions of Five-Dimensional General Relativity without Spatial Symmetry*, *Gen. Rel. Grav.* **18** (1986) 861-877.
 - [11] M. Cvetič and D. Youm, *All the Four Dimensional Static, Spherically Symmetric Solutions of Abelian Kaluza-Klein Theory*, *Phys. Rev. Lett.* **75** (1995) 4165-4168; [hep-th/9503082](#).
 - [12] J. A. Wheeler, *Geons*, *Phys. Rev.* **97** (1955) 511-536.
 - [13] M. S. Morris, K. S. Thorne and U. Yurtsever, *Wormholes, Time Machines, and the Weak Energy Condition*, *Phys. Rev. Lett.* **61** (1988) 1446-1449.
 - [14] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking*, American Institute of Physics, Woodhury, New York, 1995.
 - [15] A. Chodos and S. Detweiler, *Spherically Symmetric Solutions in Five-Dimensional General Relativity*, *Gen. Rel. Grav.* **14** (1982) 879-890.
 - [16] G. Clément, *Axisymmetric Regular Multi-Wormhole Solutions in Five-Dimensional General Relativity*, *Gen. Rel. Grav.* **16** (1984) 477-489.
 - [17] G. Clément, *A Class of Wormhole Solutions to Higher Dimensional General Relativity*, *Gen. Rel. Grav.* **16** (1984) 131-138.
 - [18] A. G. Agnese, A. P. Billyard, H. Liu and P. S. Wesson, *Possible Wormhole Solutions in (4 + 1) Gravity*, *Gen. Rel. Grav.* **31** (1999) 527-531; [gr-qc/9905037](#).
 - [19] V. Dzhunushaliev, *Multidimensional Geometrical Model of the Renormalized Electrical Charge with Splitting of the Extra Coordinates*, *Mod. Phys. Lett.* **A13** (1998) 2179-2186; [gr-qc/9807080](#).
 - [20] V. Dzhunushaliev and H.-J. Schmidt, *4D Wormhole with Signature Change in the Presence of Extra Dimensions*, *Grav. Cosmol.* **5** (1999) 187-190; [gr-qc/9908076](#).
 - [21] M. Azreg-Aïnou and G. Clément, *The Geodesic of the Kaluza-Klein Wormhole Solution*, *Gen. Rel. Grav.* **22** (1990) 1119-1133.
 - [22] V. Dzhunushaliev and D. Singleton, *Wormholes and Flux Tubes in 5D Kaluza-Klein Theory*, *Phys. Rev.* **D59** (1999) 064018; [gr-qc/9807086](#).
 - [23] M. Azreg-Aïnou, G. Clément, C. P. Constantinidis and J. C. Fabris, *Electrostatic Solutions in Kaluza-Klein Theory: Geometry and Stability*, *Grav. Cosmol.* **6** (2000) 207-218; [gr-qc/9911107](#).
 - [24] C.-M. Chen, D. V. Gal'tsov, K. Maeda and S. A. Sharakin, *SL(4, R) Generating Symmetry in Five-Dimensional Gravity Coupled to Dilaton and Three-Form*, *Phys. Lett.* **B453** (1999) 7-16; [hep-th/9901138](#).
 - [25] S. W. Hawking and G. F. R. Allis, *The Large Scale Structure of Space-Time*, *Phys. Lett.* **B453** (1979) 7-16; Cambridge University Press, (1973).
 - [26] V. V. Zhytnikov, *GRG: Computer Algebra System for Differential Geometry, Gravitation and Field Equation*, V3.2 (Moicow, 1997); GRG3.2 is available via anonymous ftp from [ftp.maths.qmw.ac.uk](#) in the directory [pub/grg3.2](#).
 - [27] C.-M. Chen, D. N. Gal'tsov and S. A. Sharakin, *Vacuum Interpretation for Supergravity M-branes*, *Phys. Lett.* **B475** (2000) 269-274; [hep-th/9908133](#).
 - [28] C.-M. Chen, D. V. Gal'tsov and S. A. Sharakin, *Einstein Gravity — Supergravity Correspondence*, [hep-th/9912127](#).

- [29] V. Dzhunushaliev and D. Singleton, *Flux Tube Solutions in Kaluza-Klein Theory*, *Class. Quant. Grav.* **16** (1999) 973-978; gr-qc/9805104.